Non Hermitian Matrix Hamiltonian and Charge Conjugation Operator

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Abstract
We obtain the charge conjugation operator and the positive definite operator for a Non-Hermitian matrix Hamiltonian and also show that this Hamiltonian is equivalent to a Hermitian Hamiltonian under a similarity transformation.

Keywords: Charge conjugation operator, Positive definite operator, Similarity transformation.

Resumen
Obtenemos el operador de conjugación de carga y el operador definido positivo de una matriz Halmitoniana No Hermítica y también mostramos que este Halmitoniano es equivalente a un Halmitoniano Hermítico bajo transformaciones de similaridad

Palabras clave: Operador de conjugación de la Carga, Operador positivo definido, Transformación de Similaridad.

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I. INTRODUCTION

PT-symmetric quantum mechanics [1] have generated much interest in recent years. The main reason for this is that the energy eigenvalues of a number of complex potentials turned out to be real (at least partly), which contradicted the usual expectations regarding non-Hermitian systems. This unusual behavior of the energy eigenvalues was attributed to the so-called PT-symmetry [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. However, a significant barrier to the physical interpretation of such theories was that the natural metric in Hilbert space H is indefinite. Recently, Bender and his co-workers [11] in their very noteworthy work have found that the class of non Hermitian Hamiltonian having an unbroken PT-symmetry also possesses a further symmetry, called complex linear operator C, similar to the charge conjugation operator. The probabilistic interpretation of quantum mechanics can be restored with the construction of new inner product using the CPT-symmetry. The PT and CPT inner product [11, 12] have been defined as

$$\langle\psi|\phi\rangle \equiv [PT\psi]^T \cdot \phi \ , \ \ \langle\psi|\phi\rangle \equiv [CPT\psi]^T \cdot \phi \qquad (1)$$

A complementary approach in constructing physically meaningful theories with non Hermitian Hamiltonian admitting real spectrum is to introduce the notion of pseudo-Hermiticity [13]. An operator is said to be pseudo-Hermitian, if it is related to its Hermitian adjoint through a similarity transformation. The non-Hermitian Hamiltonians admitting real spectrum is shown to be pseudo-Hermitian and are invariant under an anti-linear symmetry which reduces to the standard PT symmetry for some cases. Mostafazadeh has claimed that the PT inner product is just the pseudo inner product and the CPT inner product is just the η_+ inner product [14]

$$\langle\langle\psi,\phi\rangle\rangle_\eta \equiv \langle\psi,\eta\phi\rangle,\ \langle\langle\psi,\phi\rangle\rangle \equiv \langle\psi,\phi\rangle = \langle\psi,\eta\phi\rangle, \qquad (2)$$

where $\langle\psi,\phi\rangle = \psi^\Gamma.\phi,$ for all $\psi,\phi \in H$ (H =Hilbert space, "Γ" denotes Hermitian adjoint) and η_+ is positive definite operator. He has further shown that under a similarity transformation implemented by $\rho_+ = \sqrt{\eta_+}$ such a Hamiltonian is equivalent to a Hermitian Hamiltonian *h*, according to

$$h = \rho_+ H \rho_+^{-1}. \qquad (3)$$

In our present article, we give a PT-symmetric Hamiltonian and calculate the C operator. We also obtain the positive definite operator and corresponding Hermitian Hamiltonian.

The plan of the paper is as follows. In Sec. II we obtain the eigenvalue and eigenfunctions of the matrix Hamiltonian. In Sec. III, we calculate the C operator for this Hamiltonian. In Sec. IV, we obtain similarity transformation and corresponding Hermitian Hamiltonian. Finally, Sec. V, is kept for conclusions and discussions.

II. HAMILTONIAN WITH THREE FREE REAL PARAMETERS

Hamiltonians with three free real parameters have been discussed in [7, 12]. In this paper, we consider the Hamiltonian

$$H=\begin{pmatrix}\cosh_q(\alpha-i\beta) & i\sinh_q(\alpha-i\beta)\\ i\sinh_q(\alpha-i\beta) & -\cosh_q(\alpha-i\beta)\end{pmatrix}, \tag{4}$$

Where *q, α, β* are three free real parameters. Hamiltonian (4) is non Hermitian and PT-symmetric with respect to the parity

$$P=\begin{pmatrix}\cos(\beta) & \sin(\beta)\\ \sin(\beta) & -\cos(\beta)\end{pmatrix}, \tag{5}$$

and the time reversal operator act as complex conjugation. The deformed hyperbolic functions are defined by

$$\sinh_q x=\frac{e^x-qe^{-x}}{2}\ ,\ \cosh_q x=\frac{e^x+qe^{-x}}{2}\ ,$$
$$\tanh_q x=\frac{\sinh_q x}{\cosh_q x}.$$

The eigenvalues for H in (4) are

$$\varepsilon_\pm=\pm\sqrt{q}\ . \tag{6}$$

q>0 is the region of unbroken PT-symmetry. When q>0, the two linearly independent eigenvectors of H are

$$\psi_\pm=\frac{1}{\sqrt{2(1-\sqrt{q}\sec h_q\alpha)\sqrt{q}\sec h_q\alpha}}\times \begin{pmatrix}\tanh_q(\alpha)\cos\frac{\beta}{2}-i(1\mp\sqrt{q}\,\mathrm{sech}_q(\alpha))\sin\frac{\beta}{2}\\ \tanh_q(\alpha)\sin\frac{\beta}{2}+i(1\mp\sqrt{q}\,\mathrm{sech}_q(\alpha))\cos\frac{\beta}{2}\end{pmatrix}. \tag{7}$$

With respect to the inner product (1), the followings are true

$$\langle\psi_+,\psi_+\rangle=1\ ,\ \langle\psi_-,\psi_-\rangle=-1,$$
$$\langle\psi_+,\psi_-\rangle=\langle\psi_-,\psi_+\rangle=0\,. \tag{8}$$

The eigenvectors of H^Γ are $\psi_\pm^*$ since $H^\Gamma=H^*$. Setting $\phi_\pm=\pm\psi_\pm^*$, we have two linearly independent eigenvectors of

$$\phi_\pm=\pm\frac{1}{\sqrt{2(1-\sqrt{q}\sec h_q\alpha)\sqrt{q}\sec h_q\alpha}}\times \begin{pmatrix}\tanh_q(\alpha)\cos\frac{\beta}{2}+i(1\mp\sqrt{q}\sec h_q(\alpha))\sin\frac{\beta}{2}\\ \tanh_q(\alpha)\sin\frac{\beta}{2}-i(1\mp\sqrt{q}\sec h_q(\alpha))\cos\frac{\beta}{2}\end{pmatrix}. \tag{9}$$

III. C-OPERATOR

In the Hilbert space $H=C^2$, $\psi_\pm$ and $\phi_\pm$ form a complete bi-orthonormal system $\{\psi_\pm,\phi_\pm\}$ and $\psi_+.\phi_+^\Gamma+\psi_-.\phi_-^\Gamma=I$, where I is the identity matrix. The charge conjugation and positive definite operator are given by [12, 14]

$$\begin{aligned}C&=\psi_+.\phi_+^\Gamma-\psi_-.\phi_-^\Gamma\,,\\ &=\frac{1}{\sqrt{q}}\begin{pmatrix}\cosh_q(\alpha-i\beta) & i\sinh_q(\alpha-i\beta)\\ i\sinh_q(\alpha-i\beta) & -\cosh_q(\alpha-i\beta)\end{pmatrix},\\ &=\frac{1}{\sqrt{q}}H\,,\end{aligned} \tag{10}$$

$$\begin{aligned}\eta_+&=\phi_+.\phi_+^\Gamma+\phi_-.\phi_-^\Gamma\,,\\ &=\frac{1}{\sqrt{q}}\begin{pmatrix}\cosh_q(\alpha) & i\sinh_q(\alpha)\\ -i\sinh_q(\alpha) & \cosh_q(\alpha)\end{pmatrix}.\end{aligned} \tag{11}$$

One can easily verify the following results:

$$C^2=1,\ [C,PT]=0\,,\ [C,H]=0\,,\ (CP)=(CP)^*$$
$$C=\eta_+^\Gamma P\ ,\ \eta_+H\eta_+^{-1}=H^\Gamma\,.$$

With respect to the inner product (2) we have

$$\langle\psi_+,\psi_+\rangle=1,\ \ \langle\psi_-,\psi_-\rangle=1\,. \tag{12}$$

IV. SIMILARITY TRANSFORMATION

The similarity transformation ρ_+ of η_+ becomes

$$\rho_+=\frac{1}{\sqrt[4]{q}}\begin{pmatrix}\cosh_{\sqrt{q}}\left(\frac{\alpha}{2}\right) & i\sinh_{\sqrt{q}}\left(\frac{\alpha}{2}\right)\\ -i\sinh_{\sqrt{q}}\left(\frac{\alpha}{2}\right) & \cosh_{\sqrt{q}}\left(\frac{\alpha}{2}\right)\end{pmatrix}. \tag{13}$$

Inserting (13) into (3), the corresponding Hermitian Hamiltonian h is

$$h = \sqrt{q}\begin{pmatrix} \cos(\beta) & \sin(\beta) \\ \sin(\beta) & -\cos(\beta) \end{pmatrix}, \qquad (14)$$
$$= \sqrt{q}P .$$

From (10) and (14) we have $h^{-1}PH = C$.

V. CONCLUSION

In this paper, we have obtained the charge conjugation operator, positive definite operator and the corresponding Hermitian Hamiltonian. We have found that the Hamiltonian (4) is coincide with the charge conjugation operator C and the corresponding Hermitian Hamiltonian (14) is coincide with the parity operator P for q=1. We have also shown that the energy eigenvalues depend on the deformation parameter q.

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